The Nature of Chaos in a Simple Dynamical System

Akira Shibata, Toshihiro Mayuyama, Masahiro Mizutani, and Nobuhiko Saitô
Department of Applied Physics, Waseda University, Tokyo 160, Japan

Z. Naturforsch. **34a**, 1283–1289 (1979); received September 14, 1979

A simple one-dimensional transformation $x_n = a x_{n-1} + 2 - a$ $(0 \leqq x_{n-1} < 1 - 1/a)$, $x_n = a(1 - x_{n-1})$ $(1 - 1/a \leqq x_{n-1} \leqq 1)$ $(1 < a \leqq 2)$ is investigated by introducing the probability distribution function $W_n(x)$. $W_n(x)$ converges when $n \to \infty$ for $a > \sqrt{2}$, but oscillates for $1 < a \leqq \sqrt{2}$. The final distribution of $W_n(x)$ does not depend on the initial distributions for $a > \sqrt{2}$, but does for $1 < a \leqq \sqrt{2}$. Time-correlation functions are also calculated.

1. Introduction

Chaos is found sometimes in simple dynamical systems. The striking example is May's one-dimensional transformation [1],

$$x_n = G(x_{n-1}) \quad (0 \leqq x \leqq 1), \tag{1}$$

where G is a nonlinear function which has a single hump in the interval of x under consideration, such as given by $G(x) = ax(1-x)$. Similar mapping is also found in Lorenz systems [2]. Stimulated by these works, Li and Yorke demonstrated a theorem [3] which states that the existence of a solution of period three implies chaos in the sense that for each integer $n = 1, 2, \ldots$ there is a periodic point with period n and furthermore there is an uncountable subset of points x in an interval which are not even asymptotically periodic. Necessary and sufficient condition for chaos was given by Oono [4], so that period $\neq 2^n$ implies chaos. However we still do not know what is the nature of chaos, or what are the quantities which characterize the chaos.

In this paper we consider a simple example,

$$\begin{aligned} x_n = G(x_{n-1}) &= a\,x_{n-1} && (0 \leqq x_{n-1} < 1/2) \\ &= a(1 - x_{n-1}) && (1/2 \leqq x_{n-1} \leqq 1)\,, \\ & && (1 < a \leqq 2)\,. \end{aligned} \tag{2}$$

which gives rise to chaos since this system has the solution of period three for $a \geqq (1+\sqrt{5})/2$, and discuss the temporal behavior of the distribution function and the time-correlation function. The case $a = 2$ allows us an analytical treatment which will be given in Sect. 2 and 4.

Reprint requests to N. Saitô, Dept. Appl. Physics, Waseda University, Tokyo 160, Japan.

Since $1 < a \leqq 2$ in Eq. (2), $G(1/2) = a/2 > 1/2$, and $G(a/2) = a(1 - a/2) < 1/2$, it is easy to see that there exists an integer N such that for

$$\begin{aligned} &x \in [0, a(1 - a/2)) \cup (a/2, 1]\,, \\ &G^n(x) \in [a(1 - a/2), a/2] \quad \text{for all } n \geqq N, \end{aligned}$$

and that for

$$\begin{aligned} &x \in [a(1 - a/2), a/2]\,, \\ &G^m(x) \in [a(1 - a/2), a/2] \quad \text{for any } m \geqq 0\,. \end{aligned}$$

Therefore the interval $[a(1-a/2), a/2]$ absorbs the interval $[0, a(1-a/2)) \cup (a/2, 1]$. Since we consider the asymptotic behavior of $G(x)$, we may restrict $G(x)$ to the interval $[a(1-a/2), a/2]$. Thus (2) is equivalent to (3) in Sect. 2 (see Figure 1a).

2. Distribution Functions

It is well known that the Baker's transformation is isomorphic to the Bernoulli shift $B(1/2, 1/2)$, and $B(1/2, 1/2)$ coincides with a coin tossing from a stand point of the stochastic process. We consider a one-dimensional linear transformation,

$$\begin{aligned} x_n = F(x_{n-1}) &= a\,x_{n-1} + 2 - a \\ &\qquad (0 \leqq x_{n-1} < 1 - 1/a) \\ &= a(1 - x_{n-1}) \\ &\qquad (1 - 1/a \leqq x_{n-1} \leqq 1)\,, \end{aligned} \tag{3}$$

which is a projection onto the x-axis of a modified area-preserving Baker's transformation defined by

$$\begin{aligned} x_n &= a\,x_{n-1} + 2 - a \quad (0 \leqq x_{n-1} < 1 - 1/a) \\ y_n &= y_{n-1}/a\,, \\ x_n &= a(1 - x_{n-1}) \qquad (1 - 1/a \leqq x_{n-1} \leqq 1)\,, \\ y_n &= -\,y_{n-1}/a + 2^{n+1}/a^{n+1}\,, \end{aligned} \tag{4}$$

where a is a parameter $(1 < a \leqq 2)$.

0340-4811 / 79 / 1100-1283 $ 01.00/0. — Please order a reprint rather than making your own copy.

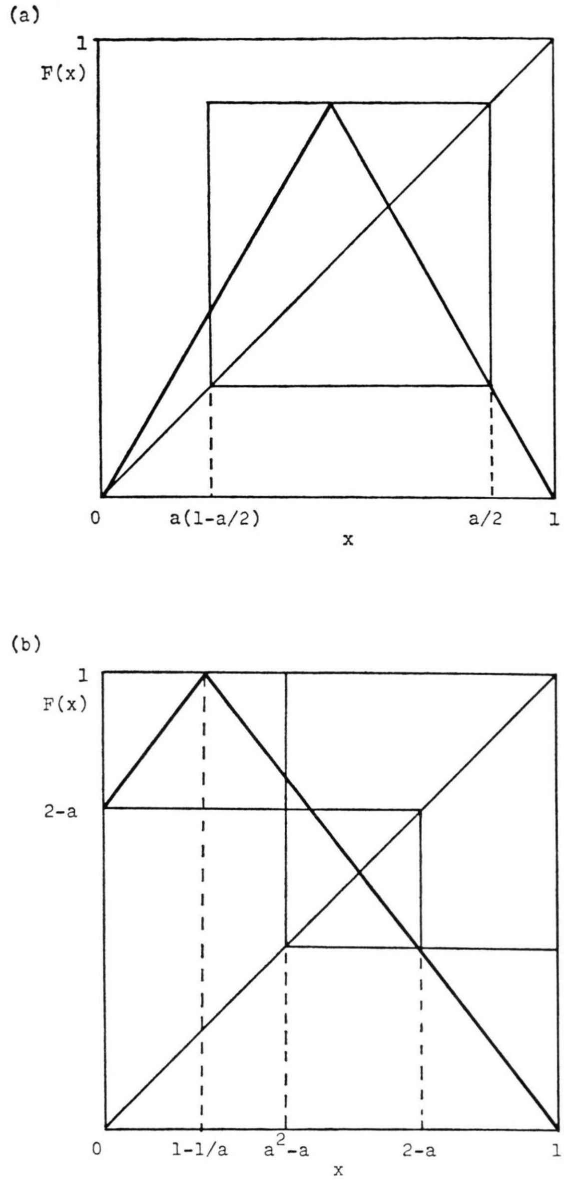

Fig. 1. Equivalent transformations of Eq. (3):
(a) $1 < a \leqq 2$, (b) $2^{1/4} < a < \sqrt{2}$.

Equation (3) determines the sequence of the values, $x_0, x_1, x_2, \ldots$, resulted by iterations of (3). We study the behavior of (3) by introducing the probability distribution (or density) function $W_n(x)$ defined in the interval [0, 1], so as to mean that the probability of finding the value x in the interval $(x, x+dx)$ is $W_n(x)\,\mathrm{d}x$. In general $W_n(x)$ for Eq. (1) obeys the equation

$$W_n(x) = LW_{n-1}(x) = \sum_{x'=G^{-1}(x)} W_{n-1}(x')/|G'(x')| , \quad (5)$$

where G: $[0, 1] \to [0, 1]$ is nonsingular and L is a linear operator, which is equivalent to the Frobenius-Perron operator [6].

In the case of (3), (5) becomes

$$\begin{aligned} W_n(x) &= LW_{n-1}(x) = 1/a\, W_{n-1}(1 - x/a) \\ &\qquad (0 \leqq x < 2 - a) \\ &= 1/a[W_{n-1}(1 - x/a) + W_{n-1}(1 - (2 - x)/a)] \\ &\qquad (2 - a \leqq x \leqq 1) . \end{aligned} \quad (6)$$

Our problem is to see the convergence, or the existence of $\lim_{n \to \infty} W_n(x)$.

In the case $a = 2$, any arbitrary initial distribution of class L^2 is shown analytically to converge to the uniform distribution for $n \to \infty$, as discussed below.

In the case $a = 2$, (6) becomes

$$W_n(x) = 1/2\,[W_{n-1}(x/2) + W_{n-1}(1 - x/2)] . \quad (7)$$

To apply the Fourier-cosine transformation to $W_n(x)$, we extend $W_n(x)$ to the interval $(-\infty, \infty)$, by assuming $W_n(-x) = W_n(x)$, and a periodic function with period 2. It is understood from (7) that if $W_{n-1}(x)$ has these properties then also $W_n(x)$ does. Therefore we can put as

$$W_n(x) = \sum_{m=0}^{\infty} A_m^{(n)} \cos(m \pi x) , \quad (8)$$

where

$$\begin{aligned} A_m^{(n)} &= \int_{-1}^{1} W_n(x) \cos(m \pi x)\, \mathrm{d}x \\ &= \int_0^1 W_{n-1}(x/2) \cos(m \pi x)\, \mathrm{d}x \\ &\quad + \int_0^1 W_{n-1}(1 - x/2) \cos(m \pi x)\, \mathrm{d}x \\ &= A_{2m}^{(n-1)} . \end{aligned} \quad (9)$$

Thus from induction we get $A_m^{(n)} = A_{2^n m}^{(0)}$. Therefore the coefficient $A_m^{(n)}$ for $m \neq 0$ tends to zero, because we have $A_k^{(o)} \to 0$ for $k \to \infty$ by virtue of the relation

$$\sum_{k=-\infty}^{\infty} |A_k^{(0)}|^2 = \int_{-1}^{1} |W_0(x)|^2\, \mathrm{d}x < \text{finite}.$$

This implies that the limiting distribution is uniform over [0, 1].

For $a < 2$, we can see that the transformation (6) gives rise to a gap at the point $2 - a$ in the interval [0, 1] for $W_1(x)$ when the function $W_0(x)$ is uniform. Successively new points of discontinuity appear when the transformation (6) proceeds further. Thus the uniform distribution cannot be the limiting distribution function, contrary to the case $a = 2$.

Now we consider the case $a \neq 2$, and take the uniform distribution for $W_0(x)$ as an initial condition and compute $W_n(x)$ through Eq. (6) (see Figure 2a). Numbers written in Fig. 2a indicate the successive points of discontinuity resulted by the iteration of (6). Notice that as the number becomes larger, the height of the gap becomes lower. This fact is also understood from (6) since the height of the new gap is multiplied by a factor of $1/a$ at each iteration. Using this fact it is understood that $W_n(x)$ converges when $n \to \infty$, provided that the gap points are not degenerate. Generally we have an infinite number of gap points of the function $\lim_{n\to\infty} W_n(x)$, which is invariant under the operator L, and is considered

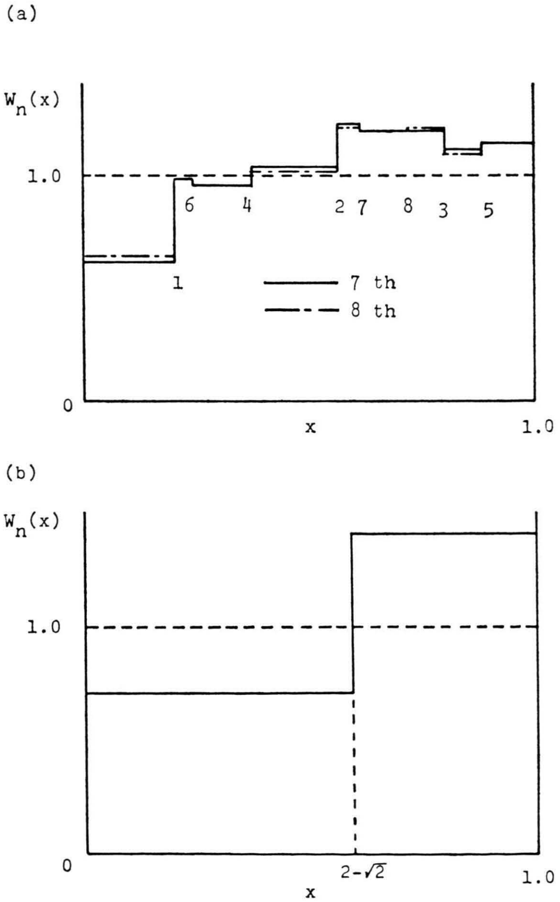

Fig. 2. Calculations of $W_n(x)$. ---- Initial condition: (a) $a = 1.8$, (b) $a = \sqrt{2}$.

as one of the invariant measures of (3). Using Ulam's approximation by step functions, Li [8] and later Grossmann et al. [6] calculated the invariant measure for broken linear transformations like (3). Li divided [0, 1] into n equal subintervals, but for our model of (3) it is adequate that we divide [0, 1] into subintervals determined by the successive gap points. Grossmann's model corresponds to the case that the gap points are degenerate. Quite recently Ito et al. [5, I] gave the same results in a series by using symbolic dynamics, and they [5, II] furthermore consider the cases with different slopes a and $-b$.

In the special cases when the gap point coincides with a peak of the transformation ($x = 1 - 1/a$) or a fixed point ($x = a/(1+a)$) at a certain number of transformations, the number of the gap points is finite. Simple examples are given; for $a = (1+\sqrt{5})/2$, a minimum value for the existence of period three, the gap point agrees with the peak by one iteration: for $a = 1.512\ldots$, a minimum value for the existence of period five, the gap point agrees with the peak by three iterations. These values of a are given by the maximal solution of the equation,

$$a^j - 2a^{j-2} - 1 = 0 \quad (j = \text{odd integer}). \tag{10}$$

When j becomes very large, a solution of (10) approaches $\sqrt{2}$ from above (see also May [1]).

Computer experiments suggest that for the value $a > \sqrt{2}$ $W_n(x)$ converges, but for $a \leqq \sqrt{2}$ the convergence breaks down suddenly.

Figure 2b ($a = \sqrt{2}$) shows that $W_n(x)$ begins to oscillate between a solid line and a dotted line when the uniform distribution is taken as the initial condition. This oscillation is understood from the fact that the interval [0, 1] is decomposed into the two intervals $[0, 2-\sqrt{2}]$ and $[2-\sqrt{2}, 1]$ and these two intervals are switching at each iteration. After two iterations the transformation (3) restricted to $[0, 2-\sqrt{2}]$ is isomorphic to the transformation (3) on the whole interval [0, 1] with $a = 2$, although the picture of the transformation is up-side-down.

Notice that the distribution function does not oscillate for all arbitrary initial distributions. To understand this fact, let $f(x)$ and $g(x)$ be the distribution functions in the interval $[0, 2-\sqrt{2}]$ and $[2-\sqrt{2}, 1]$, respectively. It is easy to see that if

$$\int_0^{2-\sqrt{2}} f(x)\,dx = \int_{2-\sqrt{2}}^{1} g(x)\,dx\,, \tag{11}$$

then the distribution function in each interval approaches the uniform distribution since the transformation with $a = \sqrt{2}$ is isomorphic with the transformation with $a = 2$. So the distribution function in the interval $[0, 1]$ becomes the limiting distribution when the above condition is satisfied.

Next we consider the case $2^{1/4} < a < 2^{1/2}$ (see Fig. 1b). In this case the interval $(a^2 - a, 2 - a)$ is absorbed into the interval $[0, a^2 - a] \cup [2 - a, 1]$ except the fixed point at certain iteration. Thus after two iterations the transformation restricted to the interval $[0, a^2 - a]$ is equivalent to the transformation in the interval $[0, 1]$ with parameter $a' = a^2$, $\sqrt{2} < a' \leqq 2$. Also the distribution function becomes the limiting distribution if the initial condition satisfies the relation

$$\int_0^{a^2-a} f(x)\,\mathrm{d}x = \int_{2-a}^{1} g(x)\,\mathrm{d}x\,, \tag{12}$$

and is zero on the interval $(a^2 - a, 2 - a)$. The same situation occurs also for $2^{1/8} < a \leqq 2^{1/4}$, and in general for $2^{1/2^n} < a \leqq 2^{1/2^{n-1}}$. Therefore the transformation (3) has a hierarchy structure and the distribution function becomes the limiting distribution in the successively decomposed 2^n intervals if the initial distribution satisfies a condition similar to those mentioned above, and if the condition is not satisfied, then the distribution function continues oscillating.

3. Temporal Behavior of $W_n(x)$ for $n \to \infty$

In this section the long-time behavior of $W_n(x)$ will be discussed. In analogy to $a = 2$ in Sect. 2, we extend $W_n(x)$ to the interval $(-\infty, \infty)$, by assuming $W_n(-x) = W_n(x)$, $W_{n-1}(-x) = W_{n-1}(x)$, and periodic functions with period 2. Therefore we can put

$$W_n(x) = \tfrac{1}{2} A_0^{(n)} + \sum_{m=1}^{\infty} A_m^{(n)} \cos(m\pi x)\,, \tag{13}$$

$$W_{n-1}(x) = \tfrac{1}{2} A_0^{(n-1)} + \sum_{m=1}^{\infty} A_m^{(n-1)} \cos(m\pi x)\,,$$

where

$$A_m^{(n)} = \int_{-1}^{1} W_n(x) \cos(m\pi x)\,\mathrm{d}x\,, \tag{14}$$

in particular

$$A_m^{(n)} = 2 \quad (n = 0, 1, 2, \ldots, n)\,.$$

Substituting (6) into (14) we obtain,

$$A_m^{(n)} = 2 \int_0^1 W_n(x) \cos(m\pi a(1 - x))\,\mathrm{d}x = \sum_{m'=0}^{\infty} B_{mm'} A_{m'}^{(n-1)}\,, \tag{15}$$

where

$$B_{m0} = \frac{\sin(m\pi a)}{m\pi a}\,, \quad B_{0m'} = \delta(m')\,, \tag{16}$$

and

$$B_{mm'} = \frac{2ma\sin(m\pi a)}{\pi(m^2a^2 - m'^2)} \quad m' \neq ma$$
$$= \cos(m\pi a) \quad m' = ma \quad (m, m' \neq 0)\,.$$

The eigenvalues of the matrix $B \equiv (B_{mm'})$ are responsible for the behavior of $A_m^{(n)}$ for $n \to \infty$. One can easily see that the matrix B has an eigenvalue 1, because $B_{0m} = \delta(m)$. To obtain the other eigenvalues we have only to consider the matrix $B' \equiv (B_{mm'}, m, m' \neq 0)$. The numerical calculations of the eigenvalues of B' are shown in Fig. 3, where the $\infty \times \infty$ matrix is approximated as 50×50 in Fig. 3a, 100×100 in Fig. 3b, and 150×150 in Fig. 3c, by ignoring $B_{mm'}$ for larger m, m' ($m, m' > 50, 100, 150$ in Fig. 3a, b, c, respectively). One finds that the eigenvalues of B' can be calculated approximately by truncating at about $m = 150$, provided that we do not consider the eigenvalues close to 0. In Fig. 4a when $a \downarrow \sqrt{2}$, we see one of the eigenvalues approaching to -1, keeping the absolute values of the others smaller than 1. The eigenvalue of -1 always exists for $a \leqq \sqrt{2}$ (Fig. 4b), from which we can understand the oscillation of $W(x)$ with period 2. Generally for $a \leqq 2^{1/2^{n-1}}$, eigenvalues λ satisfying (17) exist,

$$\lambda^{2^{n-1}} - 1 = 0 \quad (n = 1, 2, \ldots) \tag{17}$$

and other eigenvalues have absolute values smaller than 1, from which we can also understand the oscillation of $W(x)$ with period 2^{n-1}, as mentioned in the end of Sect. 2.

If we restrict ourselves to the case $\sqrt{2} < a < 2$, the largest eigenvalue of matrix B is 1 and simple. The other eigenvalues are also simple as confirmed by the computer calculations in the 150×150 matrix shown in Figure 4. Therefore we assume that the eigenvalues of B are simple. Substituting $A_m^{(n)}$ on the rhs of (13) from (15) successively, we

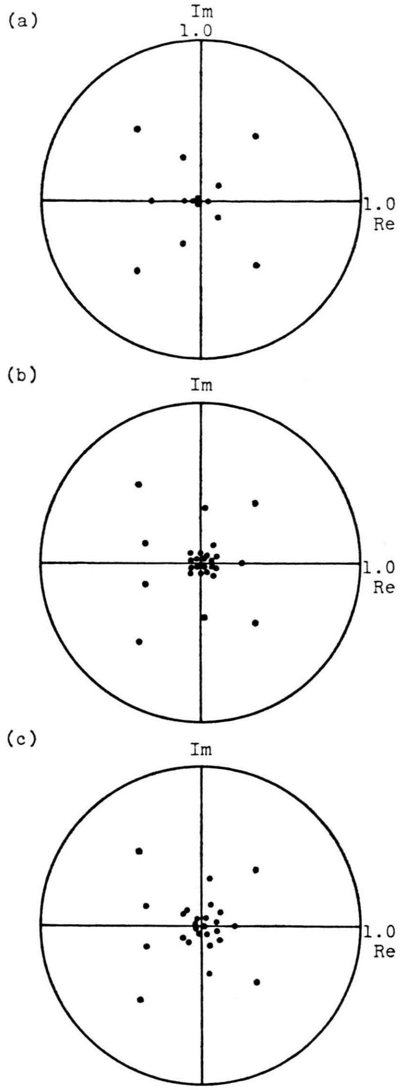

Fig. 3. Truncation of B', $a = 1.823\ldots$:
(a) $m = 50$, (b) $m = 100$, (c) $m = 150$.

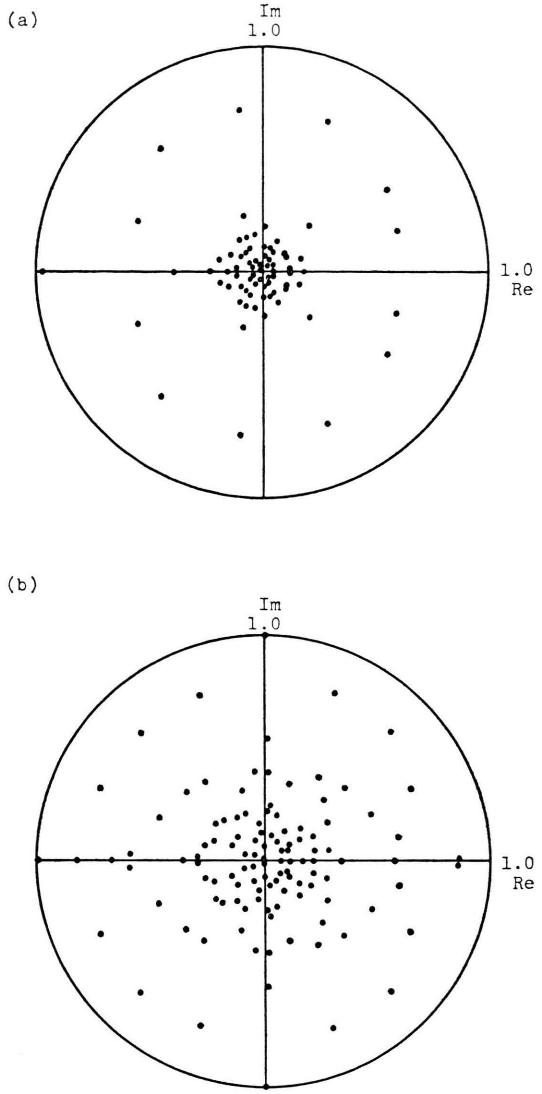

Fig. 4. Eigenvalues of B', $m = 150$.
(a) $a = 1.431\ldots > \sqrt{2}$, (b) $a = 1.180\ldots > 2^{1/8}$.

find $W_n(x)$ in the following form:

$$\begin{aligned} W_n(x) &= \mathbf{s}^{\mathrm{T}} \cdot \boldsymbol{A}^{(n)} \\ &= \mathbf{s}^{\mathrm{T}} \cdot B \cdot \boldsymbol{A}^{(n-1)} \\ &= \mathbf{s}^{\mathrm{T}} \cdot B^n \cdot \boldsymbol{A}^{(0)}\,, \end{aligned} \tag{18}$$

where $\mathbf{s}^{\mathrm{T}} \equiv (\frac{1}{2}, \cos(\pi x), \cos(2\pi x), \ldots)$ is a row vector of the transpose of $\mathbf{s}$, and

$$\boldsymbol{A}^{(n)} \equiv \begin{pmatrix} A_0^{(n)} \\ A_1^{(n)} \\ \vdots \end{pmatrix}$$

is a column vector. Since the eigenvalues of B are simple and the largest is 1, B can be diagonalized by a regular matrix V, $V^{-1} B V = \Lambda$, where $\Lambda \equiv (\lambda_m \delta_{mm'})$ is a diagonal matrix, and $V_{0m}^{-1} = \delta(m)$ since $B_{0m} = \delta(m)$. Therefore (18) becomes

$$W_n(x) = \mathbf{s}^{\mathrm{T}}\, V\, \Lambda^n\, V^{-1}\, \boldsymbol{A}^{(0)}$$
$$\xrightarrow[(n \to \infty)]{} 1 + \sum_{m=1}^{\infty} 2 \cos(m\,\pi\,x)\, V_{m0}\,. \tag{19}$$

This limiting distribution of $W_n(x)$ is independent of the initial one, and is the one obtained in Sect. 2, where the initial condition is taken as a uniform distribution.

4. Time-Correlation Functions

The properties mentioned above are also reflected into the time-correlation functions. Time-correlation functions have been calculated by Grossmann et al. [6] and Fujisaka et al. [7] for transformations similar to (3).

In this paper the correlation functions are calculated for some values of a from 2 to $\sqrt{2}$, critical values of (3). The time-correlation function $C(n)$ is defined by

$$C(n) = \langle (F^n(x) - \langle F^n(x)\rangle)(x - \langle x\rangle)\rangle / \langle (x - \langle x\rangle)^2\rangle$$
$$= (\langle F^n(x)\, x\rangle - \langle x\rangle^2)/\langle (x - \langle x\rangle)^2\rangle\,, \qquad (20)$$

where $\langle\cdots\rangle$ means the averages either by invariant measure or by long-time calculation. We take the former average for $a > \sqrt{2}$, since in this case both averages take the same value.

We can easily show that $C(n) = \delta(n)$ for $a = 2$. The numerical results of $C(n)$ for some values of a are shown in Figure 5. As the parameter a approaches $\sqrt{2}$, the correlation length becomes long. Eventually at $a = \sqrt{2}$, $C(n)$ does not decay for $n \geqq 2$. This phenomenon reflects the switching effect of two intervals as mentioned in Sect. 2.

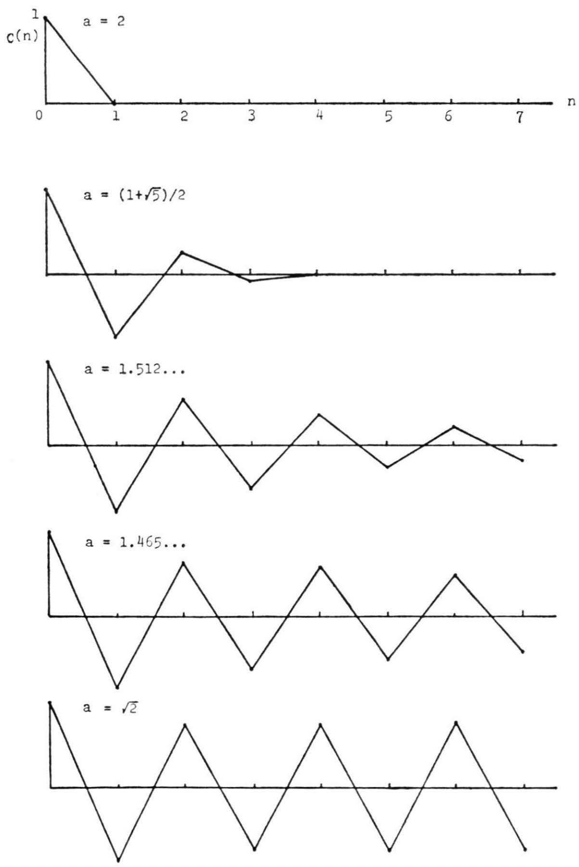

Fig. 5. Time-correlation functions.

To analyze these behaviors of the correlation functions, let us take the right and left eigenfunctions $\Psi_m^{\mathrm{R}}(x)$ and $\Psi_m^{\mathrm{L}}(x)$ of the operator L defined in (6). In accordance with (18), it is seen that

$$L(x) \equiv \int_{-1}^{1} \mathrm{d}x'\, \mathbf{s}^{\mathrm{T}}(x)\, B\, \mathbf{s}(x') \quad (\text{for } \Psi_m^{\mathrm{R}}(x))\,.$$

Therefore $\Psi_m^{\mathrm{R}}(x)$ and $\Psi_m^{\mathrm{L}}(x)$ must satisfy the relations

$$L\Psi_m^{\mathrm{R}}(x) = \lambda_m \Psi_m^{\mathrm{R}}(x)\,,$$
$$\Psi_m^{\mathrm{L}}(F(x)) = \Psi_m^{\mathrm{L}} L = \lambda_m \Psi_m^{\mathrm{L}}(x), \quad (m = 0, 1, 2, \ldots)$$
$$\int_{-1}^{1} \Psi_m^{\mathrm{L}}(x)\, \Psi_{m'}^{\mathrm{R}}(x)\, \mathrm{d}x = \delta_{mm'}\,, \qquad (21)$$

where the λ_m's are eigenvalues of B, and in particular $\lambda_0 = 1$. Generally we have the following relations:

$$\mathrm{L}g(x) = \sum_{x' = F^{-1}(x)} g(x')/|F'(x')|\,, \quad g(x)\, L = g(F(x)).$$

Now we can expand x and $F^n(x)$ by using $\Psi_m^{\mathrm{R}}(x)$ and $\Psi_m^{\mathrm{L}}(x)$,

$$x = \sum_{m=0}^{\infty} \Psi_m^{\mathrm{L}}(x) \int_0^1 x\, \Psi_m^{\mathrm{R}}(x)\, \mathrm{d}x\,,$$
$$F^n(x) = \sum_{m=0}^{\infty} \lambda_m^n\, \Psi_m^{\mathrm{L}}(x) \int_0^1 x\, \Psi_m^{\mathrm{R}}(x)\, \mathrm{d}x\,. \qquad (22)$$

The average $\langle F^n(x) x\rangle$ in (20) is expressed in the form

$$\langle F^n(x)\, x\rangle = \int_0^1 F^n(x)\, x\, \Psi_0^{\mathrm{R}}(x)\, \mathrm{d}x$$
$$= \sum_{m=0}^{\infty} \lambda_m^n\, \langle x\rangle_m\, \langle x \Psi_m^{\mathrm{R}}(x)\rangle\,, \qquad (23)$$

where

$$\langle x\rangle_m = \int_0^1 x\, \Psi_m^{R}(x)\, \mathrm{d}x\,.$$

In particular $\langle x \Psi_0^{\mathrm{L}}(x)\rangle = \langle x\rangle$, since $\Psi_0^{\mathrm{L}}(x) = 1$, i.e. $V_{0m}^{-1} = \delta(m)$. Lastly the correlation function $C(n)$ is obtained in the form

$$C(n) = \frac{\sum_{m=1}^{\infty} \lambda_m^n\, \langle x\rangle_m\, \langle x \Psi_m^{\mathrm{L}}(x)\rangle}{\sum_{m=1}^{\infty} \langle x\rangle_m\, \langle x \Psi_m^{\mathrm{L}}(x)\rangle}\,. \qquad (24)$$

For $\sqrt{2} < a < 2$, it is seen that $C(n) \to 0$ for $n \to \infty$, since the absolute value of λ_m's ($m \neq 0$) is smaller than 1. When $a \downarrow \sqrt{2}$, one of the λ_m's ($m \neq 0$) approaches -1. Therefore the length of $C(n)$ becomes long and at $a = \sqrt{2}$, $C(n)$ does not decay as seen in computer calculations.

5. Conclusions

To summarize, we have shown in the model (2) or (3) that

1. $\lim_{n\to\infty} W_n(x)$ for arbitrary initial distributions exists for $\sqrt{2} < a \leqq 2$. This limiting distribution is considered as the absolutely continuous invariant measure.
2. If we put the lowest value of a for the existence of odd period j given by (10) as a_j, we have
$$\lim_{j\to\infty} a_j = \sqrt{2} \quad (j = \text{odd integer}).$$
For $1 < a \leqq \sqrt{2}$, this system does not have odd periods, but it is in chaos according to Oono [4].
3. For $1 < a \leqq \sqrt{2}$, the distribution function $W_n(x)$ does not necessarily converge for $n \to \infty$, but it generally oscillates.
4. The correlation function has properties corresponding to the ones mentioned above of the distribution function $W_n(x)$.
5. The present method of employing the matrix B can also be applied to general one-dimensional transformations given by (1). We can expect results similar to those mentioned above for these transformations.

[1] See, as a review article, R. May, Nature London **261**, 459 (1976).
[2] E. M. Lorenz, J. Atmosph. Sci. **20**, 130 (1963).
[3] T. Y. Li and A. Yorke, Amer. Math. Monthly **82**, 985 (1975).
[4] Y. Oono, Prog. Theor. Phys. **59**, 1028–1030 (1978).
[5, I and II] S. Ito, S. Tanaka, and H. Nakada, On Unimodal linear transformation and Chaos, to appear in Tokyo J. Math.
[6] S. Grossmann and S. Thomae, Z. Naturforsch. **32a**, 1353 (1977).
[7] H. Fujisaka and T. Yamada, Z. Naturforsch. **33a**, 1455 (1978).
[8] T. Y. Li, J. Approx. Theory **17**, 177 (1976).